

$$\vec{p}_2 = M_p \vec{v}_2 = (96 \text{ kg}) (+0.38 \text{ m/s } \hat{j}) = (36.5 \text{ kg} \cdot \text{m/s}) \hat{j}.$$

Therefore, the lander's change of momentum during the first bounce is

$$\begin{aligned} \Delta \vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= (36.5 \text{ kg} \cdot \text{m/s}) \hat{j} - (-96.0 \text{ kg} \cdot \text{m/s } \hat{j}) = (133 \text{ kg} \cdot \text{m/s}) \hat{j} \end{aligned}$$

Notice how important it is to include the negative sign of the initial momentum.

Now for the comet. Since momentum of the system must be conserved, the *comet's* momentum changed by exactly the negative of this:

$$\Delta \vec{p}_c = -\Delta \vec{p} = -(133 \text{ kg} \cdot \text{m/s}) \hat{j}.$$

Therefore, its change of velocity is

$$\Delta \vec{v}_c = \frac{\Delta \vec{p}_c}{M_c} = \frac{-(133 \text{ kg} \cdot \text{m/s}) \hat{j}}{1.0 \times 10^{13} \text{ kg}} = -(1.33 \times 10^{-11} \text{ m/s}) \hat{j}.$$

Significance

This is a very small change in velocity, about a thousandth of a billionth of a meter per second. Crucially, however, it is *not* zero.



9.6 Check Your Understanding The changes of momentum for *Philae* and for Comet 67/P were equal (in magnitude). Were the impulses experienced by *Philae* and the comet equal? How about the forces? How about the changes of kinetic energies?

9.4 | Types of Collisions

Learning Objectives

By the end of this section, you will be able to:

- Identify the type of collision
- Correctly label a collision as elastic or inelastic
- Use kinetic energy along with momentum and impulse to analyze a collision

Although momentum is conserved in all interactions, not all interactions (collisions or explosions) are the same. The possibilities include:

- A single object can explode into multiple objects (one-to-many).
- Multiple objects can collide and stick together, forming a single object (many-to-one).
- Multiple objects can collide and bounce off of each other, remaining as multiple objects (many-to-many). If they do bounce off each other, then they may recoil at the same speeds with which they approached each other before the collision, or they may move off more slowly.

It's useful, therefore, to categorize different types of interactions, according to how the interacting objects move before and after the interaction.

One-to-Many

The first possibility is that a single object may break apart into two or more pieces. An example of this is a firecracker, or a bow and arrow, or a rocket rising through the air toward space. These can be difficult to analyze if the number of fragments after the collision is more than about three or four; but nevertheless, the total momentum of the system before and after the

explosion is identical.

Note that if the object is initially motionless, then the system (which is just the object) has no momentum and no kinetic energy. After the explosion, the net momentum of all the pieces of the object must sum to zero (since the momentum of this closed system cannot change). However, the system *will* have a great deal of kinetic energy after the explosion, although it had none before. Thus, we see that, although the momentum of the system is conserved in an explosion, the kinetic energy of the system most definitely is not; it increases. This interaction—one object becoming many, with an increase of kinetic energy of the system—is called an **explosion**.

Where does the energy come from? Does conservation of energy still hold? Yes; some form of potential energy is converted to kinetic energy. In the case of gunpowder burning and pushing out a bullet, chemical potential energy is converted to kinetic energy of the bullet, and of the recoiling gun. For a bow and arrow, it is elastic potential energy in the bowstring.

Many-to-One

The second possibility is the reverse: that two or more objects collide with each other and stick together, thus (after the collision) forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of objects remains constant, the kinetic energy doesn't; but this time, the kinetic energy decreases. This type of collision is called **inelastic**.

In the extreme case, multiple objects collide, stick together, and remain motionless after the collision. Since the objects are all motionless after the collision, the final kinetic energy is also zero; the loss of kinetic energy is a maximum. Such a collision is said to be **perfectly inelastic**.

Many-to-Many

The extreme case on the other end is if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called **elastic**.

In any interaction of a closed system of objects, the total momentum of the system is conserved ($\vec{p}_f = \vec{p}_i$) but the kinetic energy may not be:

- If $0 < K_f < K_i$, the collision is inelastic.
- If $K_f = 0$, the collision is perfectly inelastic.
- If $K_f = K_i$, the collision is elastic.
- If $K_f > K_i$, the interaction is an explosion.

The point of all this is that, in analyzing a collision or explosion, you can use both momentum and kinetic energy.

Problem-Solving Strategy: Collisions

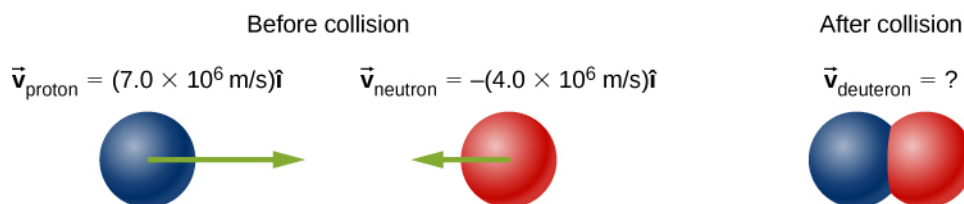
A closed system always conserves momentum; it might also conserve kinetic energy, but very often it doesn't. Energy-momentum problems confined to a plane (as ours are) usually have two unknowns. Generally, this approach works well:

1. Define a closed system.
2. Write down the expression for conservation of momentum.
3. If kinetic energy is conserved, write down the expression for conservation of kinetic energy; if not, write down the expression for the change of kinetic energy.
4. You now have two equations in two unknowns, which you solve by standard methods.

Example 9.10

Formation of a Deuteron

A proton (mass 1.67×10^{-27} kg) collides with a neutron (with essentially the same mass as the proton) to form a particle called a *deuteron*. What is the velocity of the deuteron if it is formed from a proton moving with velocity 7.0×10^6 m/s to the left and a neutron moving with velocity 4.0×10^6 m/s to the right?



Strategy

Define the system to be the two particles. This is a collision, so we should first identify what kind. Since we are told the two particles form a single particle after the collision, this means that the collision is perfectly inelastic. Thus, kinetic energy is not conserved, but momentum is. Thus, we use conservation of energy to determine the final velocity of the system.

Solution

Treat the two particles as having identical masses M . Use the subscripts p, n, and d for proton, neutron, and deuteron, respectively. This is a one-dimensional problem, so we have

$$Mv_p - Mv_n = 2Mv_d.$$

The masses divide out:

$$\begin{aligned} v_p - v_n &= 2v_d \\ 7.0 \times 10^6 \text{ m/s} - 4.0 \times 10^6 \text{ m/s} &= 2v_d \\ v_d &= 1.5 \times 10^6 \text{ m/s.} \end{aligned}$$

The velocity is thus $\vec{v}_d = (1.5 \times 10^6 \text{ m/s})\hat{i}$.

Significance

This is essentially how particle colliders like the Large Hadron Collider work: They accelerate particles up to very high speeds (large momenta), but in opposite directions. This maximizes the creation of so-called “daughter particles.”

Example 9.11

Ice Hockey 2

(This is a variation of an earlier example.)

Two ice hockey pucks of different masses are on a flat, horizontal hockey rink. The red puck has a mass of 15 grams, and is motionless; the blue puck has a mass of 12 grams, and is moving at 2.5 m/s to the left. It collides with the motionless red puck (**Figure 9.20**). If the collision is perfectly elastic, what are the final velocities of the two pucks?

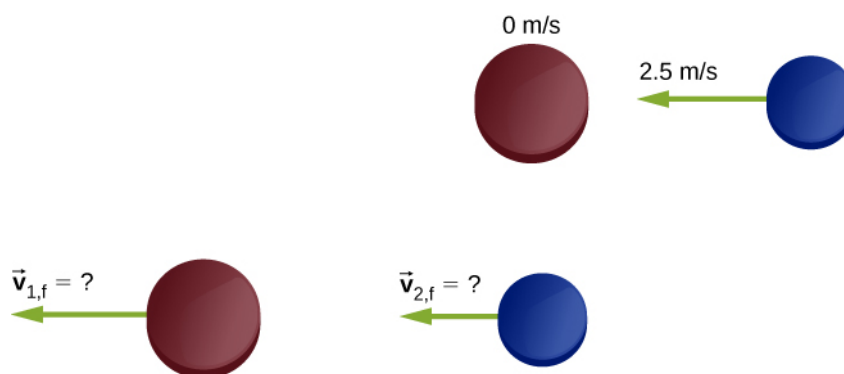


Figure 9.20 Two different hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, and we're told their masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy; define the system to be the two pucks. There is no friction, so we have a closed system. We have two unknowns (the two final velocities), but only one equation. The comment about the collision being perfectly elastic is the clue; it suggests that kinetic energy is also conserved in this collision. That gives us our second equation.

The initial momentum and initial kinetic energy of the system resides entirely and only in the second puck (the blue one); the collision transfers some of this momentum and energy to the first puck.

Solution

Conservation of momentum, in this case, reads

$$p_i = p_f$$

$$m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}.$$

Conservation of kinetic energy reads

$$K_i = K_f$$

$$\frac{1}{2}m_2 v_{2,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2.$$

There are our two equations in two unknowns. The algebra is tedious but not terribly difficult; you definitely should work it through. The solution is

$$v_{1,f} = \frac{(m_1 - m_2)v_{1,i} + 2m_2 v_{2,i}}{m_1 + m_2}$$

$$v_{2,f} = \frac{(m_2 - m_1)v_{2,i} + 2m_1 v_{1,i}}{m_1 + m_2}.$$

Substituting the given numbers, we obtain

$$v_{1,f} = 2.22 \frac{\text{m}}{\text{s}}$$

$$v_{2,f} = -0.28 \frac{\text{m}}{\text{s}}.$$

Significance

Notice that after the collision, the blue puck is moving to the right; its direction of motion was reversed. The red puck is now moving to the left.



9.7 Check Your Understanding There is a second solution to the system of equations solved in this example (because the energy equation is quadratic): $v_{1,f} = -2.5 \text{ m/s}$, $v_{2,f} = 0$. This solution is unacceptable on physical grounds; what's wrong with it?

Example 9.12

Thor vs. Iron Man

The 2012 movie “The Avengers” has a scene where Iron Man and Thor fight. At the beginning of the fight, Thor throws his hammer at Iron Man, hitting him and throwing him slightly up into the air and against a small tree, which breaks. From the video, Iron Man is standing still when the hammer hits him. The distance between Thor and Iron Man is approximately 10 m, and the hammer takes about 1 s to reach Iron Man after Thor releases it. The tree is about 2 m behind Iron Man, which he hits in about 0.75 s. Also from the video, Iron Man's trajectory to the tree is very close to horizontal. Assuming Iron Man's total mass is 200 kg:

- Estimate the mass of Thor's hammer
- Estimate how much kinetic energy was lost in this collision

Strategy

After the collision, Thor's hammer is in contact with Iron Man for the entire time, so this is a perfectly inelastic collision. Thus, with the correct choice of a closed system, we expect momentum is conserved, but not kinetic energy. We use the given numbers to estimate the initial momentum, the initial kinetic energy, and the final kinetic energy. Because this is a one-dimensional problem, we can go directly to the scalar form of the equations.

Solution

- First, we posit conservation of momentum. For that, we need a closed system. The choice here is the system (hammer + Iron Man), from the time of collision to the moment just before Iron Man and the hammer hit the tree. Let:
 - M_H = mass of the hammer
 - M_I = mass of Iron Man
 - v_H = velocity of the hammer before hitting Iron Man
 - v = combined velocity of Iron Man + hammer after the collision

Again, Iron Man's initial velocity was zero. Conservation of momentum here reads:

$$M_H v_H = (M_H + M_I)v.$$

We are asked to find the mass of the hammer, so we have

$$\begin{aligned} M_H v_H &= M_H v + M_I v \\ M_H (v_H - v) &= M_I v \\ M_H &= \frac{M_I v}{v_H - v} \\ &= \frac{(200 \text{ kg})\left(\frac{2 \text{ m}}{0.75 \text{ s}}\right)}{10 \frac{\text{m}}{\text{s}} - \left(\frac{2 \text{ m}}{0.75 \text{ s}}\right)} \\ &= 73 \text{ kg}. \end{aligned}$$

Considering the uncertainties in our estimates, this should be expressed with just one significant figure; thus, $M_H = 7 \times 10^1 \text{ kg}$.

- The initial kinetic energy of the system, like the initial momentum, is all in the hammer:

$$\begin{aligned}
 K_i &= \frac{1}{2} M_H v_H^2 \\
 &= \frac{1}{2} (70 \text{ kg})(10 \text{ m/s})^2 \\
 &= 3500 \text{ J}.
 \end{aligned}$$

After the collision,

$$\begin{aligned}
 K_f &= \frac{1}{2} (M_H + M_I) v^2 \\
 &= \frac{1}{2} (70 \text{ kg} + 200 \text{ kg})(2.67 \text{ m/s})^2 \\
 &= 960 \text{ J}.
 \end{aligned}$$

Thus, there was a loss of $3500 \text{ J} - 960 \text{ J} = 2540 \text{ J}$.

Significance

From other scenes in the movie, Thor apparently can control the hammer's velocity with his mind. It is possible, therefore, that he mentally causes the hammer to maintain its initial velocity of 10 m/s while Iron Man is being driven backward toward the tree. If so, this would represent an external force on our system, so it would not be closed. Thor's mental control of his hammer is beyond the scope of this book, however.

Example 9.13

Analyzing a Car Crash

At a stoplight, a large truck (3000 kg) collides with a motionless small car (1200 kg). The truck comes to an instantaneous stop; the car slides straight ahead, coming to a stop after sliding 10 meters. The measured coefficient of friction between the car's tires and the road was 0.62. How fast was the truck moving at the moment of impact?

Strategy

At first it may seem we don't have enough information to solve this problem. Although we know the initial speed of the car, we don't know the speed of the truck (indeed, that's what we're asked to find), so we don't know the initial momentum of the system. Similarly, we know the final speed of the truck, but not the speed of the car immediately after impact. The fact that the car eventually slid to a speed of zero doesn't help with the final momentum, since an external friction force caused that. Nor can we calculate an impulse, since we don't know the collision time, or the amount of time the car slid before stopping. A useful strategy is to impose a restriction on the analysis.

Suppose we define a system consisting of just the truck and the car. The momentum of this system isn't conserved, because of the friction between the car and the road. But if we *could* find the speed of the car the instant after impact—before friction had any measurable effect on the car—then we could consider the momentum of the system to be conserved, with that restriction.

Can we find the final speed of the car? Yes; we invoke the work-kinetic energy theorem.

Solution

First, define some variables. Let:

- M_C and M_T be the masses of the car and truck, respectively
- $v_{T,i}$ and $v_{T,f}$ be the velocities of the truck before and after the collision, respectively
- $v_{C,i}$ and $v_{C,f}$ be the velocities of the car before and after the collision, respectively
- K_i and K_f be the kinetic energies of the car immediately after the collision, and after the car has stopped sliding (so $K_f = 0$).

- d be the distance the car slides after the collision before eventually coming to a stop.

Since we actually want the initial speed of the truck, and since the truck is not part of the work-energy calculation, let's start with conservation of momentum. For the car + truck system, conservation of momentum reads

$$p_i = p_f$$

$$M_c v_{c,i} + M_T v_{T,i} = M_c v_{c,f} + M_T v_{T,f}.$$

Since the car's initial velocity was zero, as was the truck's final velocity, this simplifies to

$$v_{T,i} = \frac{M_c}{M_T} v_{c,f}.$$

So now we need the car's speed immediately after impact. Recall that

$$W = \Delta K$$

where

$$\begin{aligned}\Delta K &= K_f - K_i \\ &= 0 - \frac{1}{2} M_c v_{c,f}^2.\end{aligned}$$

Also,

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = Fd\cos\theta.$$

The work is done over the distance the car slides, which we've called d . Equating:

$$Fd\cos\theta = -\frac{1}{2} M_c v_{c,f}^2.$$

Friction is the force on the car that does the work to stop the sliding. With a level road, the friction force is

$$F = \mu_k M_c g.$$

Since the angle between the directions of the friction force vector and the displacement d is 180° , and $\cos(180^\circ) = -1$, we have

$$-(\mu_k M_c g)d = -\frac{1}{2} M_c v_{c,f}^2$$

(Notice that the car's mass divides out; evidently the mass of the car doesn't matter.)

Solving for the car's speed immediately after the collision gives

$$v_{c,f} = \sqrt{2\mu_k g d}.$$

Substituting the given numbers:

$$\begin{aligned}v_{c,f} &= \sqrt{2(0.62)\left(9.81 \frac{\text{m}}{\text{s}^2}\right)(10 \text{ m})} \\ &= 11.0 \text{ m/s}.\end{aligned}$$

Now we can calculate the initial speed of the truck:

$$v_{T,i} = \left(\frac{1200 \text{ kg}}{3000 \text{ kg}}\right)(11.0 \frac{\text{m}}{\text{s}}) = 4.4 \text{ m/s}.$$

Significance

This is an example of the type of analysis done by investigators of major car accidents. A great deal of legal and financial consequences depend on an accurate analysis and calculation of momentum and energy.



9.8 Check Your Understanding Suppose there had been no friction (the collision happened on ice); that would make μ_k zero, and thus $v_{c,f} = \sqrt{2\mu_k g d} = 0$, which is obviously wrong. What is the mistake in this conclusion?

Subatomic Collisions and Momentum

Conservation of momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

At the beginning of the twentieth century, there was considerable interest in, and debate about, the structure of the atom. It was known that atoms contain two types of electrically charged particles: negatively charged electrons and positively charged protons. (The existence of an electrically neutral particle was suspected, but would not be confirmed until 1932.) The question was, how were these particles arranged in the atom? Were they distributed uniformly throughout the volume of the atom (as J.J. Thomson proposed), or arranged at the corners of regular polygons (which was Gilbert Lewis' model), or rings of negative charge that surround the positively charged nucleus—rather like the planetary rings surrounding Saturn (as suggested by Hantaro Nagaoka), or something else?

The New Zealand physicist Ernest Rutherford (along with the German physicist Hans Geiger and the British physicist Ernest Marsden) performed the crucial experiment in 1909. They bombarded a thin sheet of gold foil with a beam of high-energy (that is, high-speed) alpha-particles (the nucleus of a helium atom). The alpha-particles collided with the gold atoms, and their subsequent velocities were detected and analyzed, using conservation of momentum and conservation of energy.

If the charges of the gold atoms were distributed uniformly (per Thomson), then the alpha-particles should collide with them and nearly all would be deflected through many angles, all small; the Nagaoka model would produce a similar result. If the atoms were arranged as regular polygons (Lewis), the alpha-particles would deflect at a relatively small number of angles.

What *actually* happened is that nearly *none* of the alpha-particles were deflected. Those that were, were deflected at large angles, some close to 180° —those alpha-particles reversed direction completely (**Figure 9.21**). None of the existing atomic models could explain this. Eventually, Rutherford developed a model of the atom that was much closer to what we now have—again, using conservation of momentum and energy as his starting point.

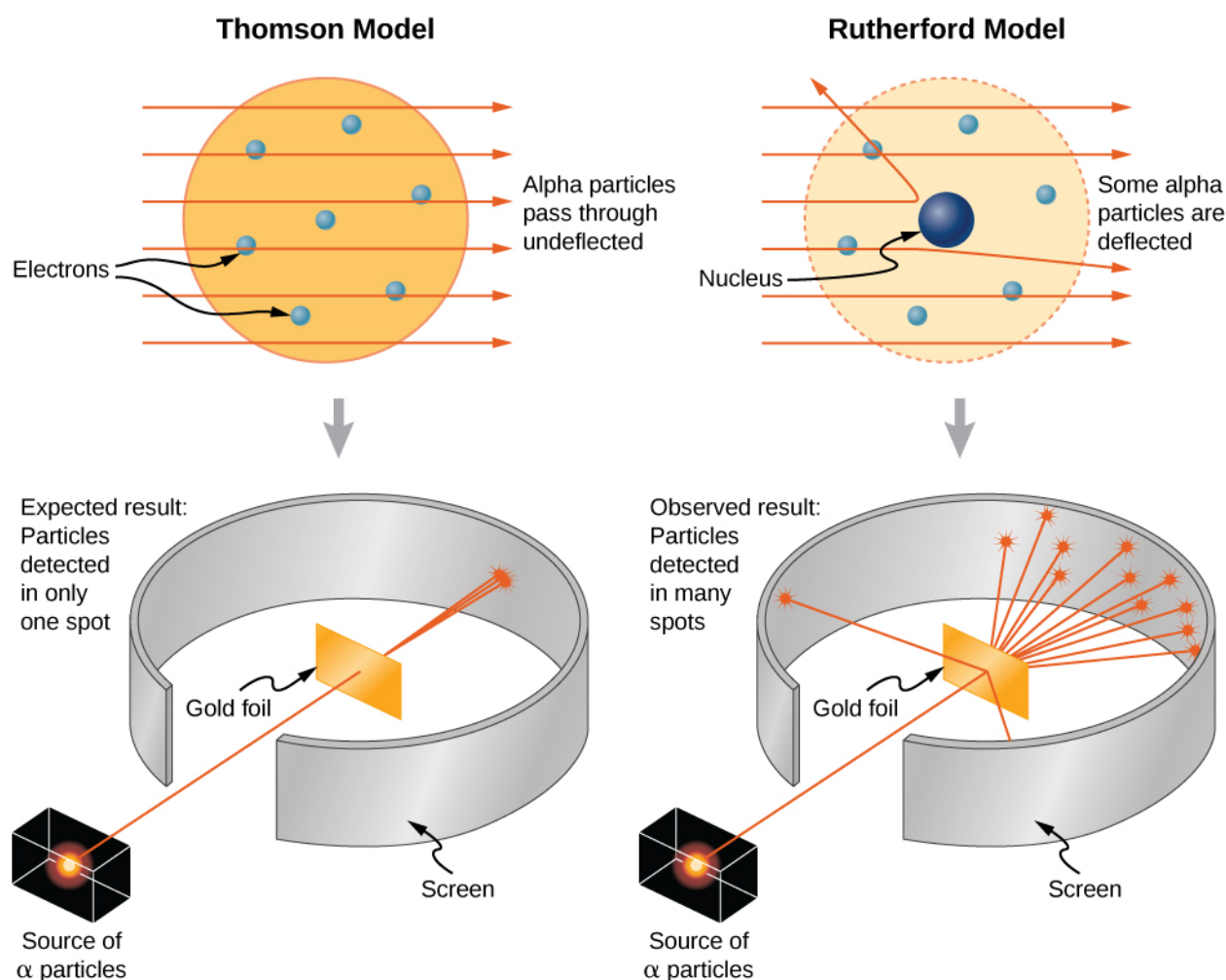


Figure 9.21 The Thomson and Rutherford models of the atom. The Thomson model predicted that nearly all of the incident alpha-particles would be scattered and at small angles. Rutherford and Geiger found that nearly none of the alpha particles were scattered, but those few that were deflected did so through very large angles. The results of Rutherford's experiments were inconsistent with the Thomson model. Rutherford used conservation of momentum and energy to develop a new, and better model of the atom—the nuclear model.

9.5 | Collisions in Multiple Dimensions

Learning Objectives

By the end of this section, you will be able to:

- Express momentum as a two-dimensional vector
- Write equations for momentum conservation in component form
- Calculate momentum in two dimensions, as a vector quantity

It is far more common for collisions to occur in two dimensions; that is, the angle between the initial velocity vectors is neither zero nor 180° . Let's see what complications arise from this.

The first idea we need is that momentum is a vector; like all vectors, it can be expressed as a sum of perpendicular components (usually, though not always, an x-component and a y-component, and a z-component if necessary). Thus, when we write down the statement of conservation of momentum for a problem, our momentum vectors can be, and usually will be, expressed in component form.

The second idea we need comes from the fact that momentum is related to force: